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Differential geometry of the vortex filament equation

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Abstract

Differential calculus on the space of asymptotically linear curves is developed. The calculus is applied to elucidate the integrability of the vortex filament equation.

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1. Introduction

The vortex filament equation [1] is a nonlinear evolution equation describing the time development of a very thin vortex tube. The equation is derived from the dynamics of three-dimensional incompressible fluid with the local induction approximation and is written as

$$\dot{\gamma} = \kappa \mathbf{b}, \tag{1}$$

where γ is the curve of the vortex filament parametrized by the arclength, dot stands for the differential with respect to the time, κ is the curvature of γ, and $\mathbf{b}$ is the bi-normal vector along γ. It is well known that the vortex filament equation (1) is closely related to the cubic nonlinear Schrödinger (NLS) equation, and the Hasimoto map provides a connection between them [2]. The NLS equation is an infinite-dimensional completely integrable Hamiltonian system [3].

Marsden and Weinstein [4] constructed a Hamiltonian description of the vortex filament equation in their study on the moment map for the action of the unimodular diffeomorphism

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group of $\mathbb{R}^3$. Langer and Perline [5] introduced the space BAL – the space of balanced asymptotically linear curves (see Section 2) – as a phase space for the system of vortex filament, and showed that the Hasimoto map is a Poisson map from BAL with the Marsden–Weinstein Poisson structure to (a certain equivalence class of) the phase space of the NLS system with the 'fourth' Poisson structure. This result says that the Hasimoto map induces constants of motion in involution for the vortex filament equation as the pull-back of those for the NLS system, hence the vortex filament equation can be understood as a completely integrable system. Further, Langer and Perline found a recursion operator, which generates infinite sequence of commuting Hamiltonian vector fields on BAL.

For some typical integrable Hamiltonian systems, such as the NLS equation, the integrability is studied from various aspects and many remarkable structures are known to exist [3,6–13]. It is therefore natural to ask whether the same or similar structures exist for the system of vortex filament. In this paper we focus on structures that are described in the language of differential geometry. For this aim, we introduce a differential calculus on BAL in an algebraic manner. Extensively applying the calculus, we prove the hereditary property [9,11] of the recursion operator, Hamiltonian pair [7,8], and master symmetries [10]. The asymptotic boundary condition defining BAL is critical for this result; a different situation is encountered when the curve of a vortex filament is supposed to be a loop [14].

The paper is organized as follows: In Section 2, the definition of BAL is clarified. It involves introducing a further condition to the conditions defining BAL of [5]. Also in this section, some basic notions are described, and several useful formulae for the variational calculus on BAL are summarized. In Section 3, carefully specifing what are admissible vector fields and what are admissible functionals, we define a differential calculus on BAL. In Section 4, several structures related to the integrability are investigated.

2. Balanced asymptotically linear curves

Let APC be the space of infinitely extended, arclength-parametrized smooth curves in the Euclidean space $\mathbb{R}^3$ with the standard metric $\langle\ ,\ \rangle$. We imply by the letter γ an element of APC and by s the parameter for it; $s \mapsto \gamma(s)$ is a smooth map $\mathbb{R} \to \mathbb{R}^3$ such that $\partial\gamma(s)/\partial s$ is a unit vector in the tangent space $T_{\gamma(s)}\mathbb{R}^3$.

A map APC $\times\, \mathbb{R} \to \mathbb{R}$ is referred to as a scalar field on APC. A map $\mathbf{x}$: APC $\times\, \mathbb{R} \to \coprod T_{\gamma(s)}\mathbb{R}^3$ such that $\mathbf{x}(\gamma, s) \in T_{\gamma(s)}\mathbb{R}^3$ is referred to as a tangent field (the term 'vector field' is reserved for the differential calculus). Here, $\coprod$ stands for the direct-sum with respect to the index $(\gamma, s) \in$ APC $\times\, \mathbb{R}$. Similar terminology is used also for a subset BAL, a space that we wish to manifest in this section.

The differential operator with respect to s is denoted by ∂_s (when acting on scalar fields) or by ∇_s (when acting on tangent fields). These operators satisfy

$$\partial_s(fg) = (\partial_s f)g + f\partial_s g, \tag{2}$$

$$\nabla_s(f\mathbf{x}) = (\partial_s f)\mathbf{x} + f\nabla_s\mathbf{x}, \quad \partial_s\langle\mathbf{x}, \mathbf{y}\rangle = \langle\nabla_s\mathbf{x}, \mathbf{y}\rangle + \langle\mathbf{x}, \nabla_s\mathbf{y}\rangle \tag{3}$$

for all scalar fields f, g and tangent fields $\mathbf{x}, \mathbf{y}$. As in the equations above, we will often surpress the argument (γ, s).

A scalar field F is called a functional if F is independent of s, i.e., $\partial_s F = 0$.

We say a scalar field f is asymptotically polynomial-like if there exists a polynomial $P(s) \in \mathbb{R}[s]$ such that $f(\gamma, s)/P(s) \to 0$ in the limit $s \to \pm\infty$ for every curve γ. We say a scalar field f is rapidly decreasing if $f(\gamma, s)P(s)$ for every polynomial $P(s) \in \mathbb{R}[s]$ converges to zero in the limit $s \to \pm\infty$ for every curve γ.

Let f be a scalar field. The scalar field $\partial_s^{-1} f$ (anti-differentiation of f) and the functional $\int f$ (definite integration of f) are defined by

$$(\partial_s^{-1} f)(\gamma, s) = \frac{1}{2}\left(\int_{-\infty}^{s} f(\gamma, \tilde{s})\, \mathrm{d}\tilde{s} - \int_{s}^{\infty} f(\gamma, \tilde{s})\, \mathrm{d}\tilde{s}\right), \tag{4}$$

$$\left(\int f\right)(\gamma) = \int_{-\infty}^{\infty} f(\gamma, \tilde{s})\, \mathrm{d}\tilde{s} \tag{5}$$

provided that the integrations in the equations above converge. In employing operators ∂_s^{-1} and $\int$ in the following sections, we will ensure the convergence by introducing certain rules. It is easy to see

$$\partial_s \partial_s^{-1} f = \partial_s^{-1} \partial_s f = f, \quad \partial_s \int f = \int \partial_s f = 0, \tag{6}$$

$$\partial_s^{-1}(Ff) = F \partial_s^{-1} f, \quad \int (Ff) = F \int f, \tag{7}$$

where f is a rapidly decreasing scalar field and F is a functional.

We denote by $\mathbf{t}, \mathbf{n}, \mathbf{b}$ the tangent fields forming the Frénet frame, namely, $\mathbf{t}(\gamma, s)$, $\mathbf{n}(\gamma, s)$ and $\mathbf{b}(\gamma, s)$ are orthonormal vectors in $T_{\gamma(s)}\mathbb{R}^3$ satisfying $\mathbf{t}(\gamma, s) = \partial\gamma(s)/\partial s$ and the Frénet–Serret relation

$$\nabla_s \mathbf{t} = \kappa \mathbf{n}, \quad \nabla_s \mathbf{n} = -\kappa \mathbf{t} + \tau \mathbf{b}, \quad \nabla_s \mathbf{b} = -\tau \mathbf{n}. \tag{8}$$

Here, κ and τ are scalar fields characterized by (8), namely, $\kappa(\gamma)$ and $\tau(\gamma)$ are the curvature and torsion, respectively, of the curve γ. Every tangent field is uniquely written as a linear combination of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ with the coefficients in scalar fields.

The space BAL introduced in [5] is a subset of APC such that (a) the curvature $\kappa(\gamma)$ of $\gamma \in$ BAL is non-vanishing, (b) $\gamma \in$ BAL is asymptotic to a fixed line, e.g., to z-axis, and (c) ambiguity in the parametrization is completely eliminated with imposing balancing condition.

To describe the balancing condition, we need to fix a reference curve $\gamma_0 \in$ APC (or a reference line, z-axis) fulfilling the asymptotic condition as in (b). The condition (b) says the existence of functionals $l_\pm\colon BAL \to \mathbb{R}$ with which the asymptotic behaviour of $\gamma \in$ BAL in the region $s \to \pm\infty$ is written as $\gamma(s \pm l_\pm(\gamma)) \to \gamma_0(s)$. With these functionals, the balancing condition (c) for $\gamma \in$ BAL can be written as $l_+(\gamma) = l_-(\gamma)$. The functional

$l := l_+ + l_-$ referred to as the renormalized length (relative to γ_0) is well-defined, though the curve $\gamma \in \mathrm{BAL} \subset \mathrm{APC}$ is of infinite length.

We supplement the condition (b) with prescribing how $\gamma \in \mathrm{BAL}$ converges to the reference curve γ_0; we suppose

$$\kappa \text{ is rapidly decreasing, and}$$
$$\kappa^{-1}\partial_s{}^n\kappa \text{ and } \partial_s{}^n\tau,\ n = 0, 1, \ldots \text{ are all asymptotically polynomial-like,} \tag{9}$$

where $\kappa^{-1} := 1/\kappa$.

In the next section we introduce a differential calculus on BAL. The action of vector fields on functionals in this calculus is defined to reproduce the usual variational calculus. Here we make a few remarks on the variational calculus and give several useful formulae. For more detailed description, we refer to [5].

Let $\mathbf{x}$ be a tangent field written as

$$\mathbf{x} = \partial_s^{-1}(\kappa g)\mathbf{t} + g\mathbf{n} + h\mathbf{b} \tag{10}$$

with certain rapidly decreasing scalar fields g, h. Below, $\mathbf{x}(\gamma)$ is identified with a variational vector field along γ. The restriction on $\mathbf{x}$ mentioned above is to force the variation to keep the arclength-parametrization and conditions (b) and (c).

In this paper the variational differential operator associated with $\mathbf{x}$ of the form (10) is denoted by $\delta_\mathbf{x}$ (when acting on scalar fields) or by $\nabla_\mathbf{x}$ (when acting on tangent fields). For calculating the former, the following formulae are useful:

$$\delta_\mathbf{x}(fg) = (\delta_\mathbf{x} f)g + f\delta_\mathbf{x} g, \tag{11}$$

$$\delta_\mathbf{x}\,\partial_s f = \partial_s\delta_\mathbf{x} f, \quad \delta_\mathbf{x}\,\partial_s^{-1} f = \partial_s^{-1}\delta_\mathbf{x} f, \quad \delta_\mathbf{x}\int f = \int\delta_\mathbf{x} f, \tag{12}$$

$$\delta_\mathbf{x}\,\kappa = \langle \mathbf{n}, \nabla_s\nabla_s\mathbf{x}\rangle, \tag{13}$$

$$\delta_\mathbf{x}\,\tau = \partial_s\langle \kappa^{-1}\mathbf{b}, \nabla_s\nabla_s\mathbf{x}\rangle + \langle \kappa\mathbf{b}, \nabla_s\mathbf{x}\rangle, \tag{14}$$

$$\delta_\mathbf{x}\, s = 0, \tag{15}$$

$$\delta_\mathbf{x} l = \int\langle -\kappa\mathbf{n}, \mathbf{x}\rangle, \tag{16}$$

where f and g are scalar fields. By abuse of notation, we often use the letter s, by which we mean the scalar field $\hat{s}$ such that $\hat{s}(\gamma, s) = s$, as we have done in (15). The latter can be calculated by the following formulae:

$$\nabla_\mathbf{x}(f\mathbf{y}) = (\delta_\mathbf{x} f)\mathbf{y} + f\nabla_\mathbf{x}\mathbf{y}, \tag{17}$$

$$\nabla_\mathbf{x}\mathbf{t} = \langle \mathbf{n}, \nabla_s\mathbf{x}\rangle\mathbf{n} + \langle \mathbf{b}, \nabla_s\mathbf{x}\rangle\mathbf{b}, \tag{18}$$

$$\nabla_\mathbf{x}\mathbf{n} = \langle \kappa^{-1}\mathbf{b}, \nabla_s\nabla_s\mathbf{x}\rangle\mathbf{b} - \langle \mathbf{n}, \nabla_s\mathbf{x}\rangle\mathbf{t}, \tag{19}$$

$$\nabla_\mathbf{x}\mathbf{b} = -\langle \mathbf{b}, \nabla_s\mathbf{x}\rangle\mathbf{t} - \langle \kappa^{-1}\mathbf{b}, \nabla_s\nabla_s\mathbf{x}\rangle\mathbf{n}, \tag{20}$$

where f is a scalar field and $\mathbf{y}$ is a tangent field. These satisfy

$$\delta_{\mathbf{x}} \langle \mathbf{y}, \mathbf{z} \rangle = \langle \nabla_{\mathbf{x}} \mathbf{y}, \mathbf{z} \rangle + \langle \mathbf{y}, \nabla_{\mathbf{x}} \mathbf{z} \rangle. \tag{21}$$

3. Differential calculus

Let A_n, $n \in \mathbb{Z}$, be the ∂_s-invariant space (i.e., $\partial_s f \in A_n$ for all $f \in A_n$) of scalar fields f such that $\kappa^{-n} f$ is asymptotically polynomial-like. The elements of A_n with $n > 0$ are rapidly decreasing.

We notice the following properties possessed by A_n:

(a1) $A_n, \forall n \in \mathbb{Z}$, is an $\mathbb{R}$-vector space,

(a2) $A_n \subset A_{n-1}$ (as $\mathbb{R}$-vector spaces) for all $n \in \mathbb{Z}$,

(a3) $A_{-\infty} := A_0 \cup A_{-1} \cup \cdots$ is a commutative associative $\mathbb{R}$-algebra with the unit 1,

(a4) $\{fg \in A_{-\infty} \mid f \in A_i, g \in A_j\}$ is a subspace of A_{i+j} for all $i, j \in \mathbb{Z}$,

(a5) ∂_s is an $\mathbb{R}$-linear operator such that $\partial_s(A_n) \subset A_n$ for all $n \in \mathbb{Z}$,

(a6) ∂_s^{-1} and $\int$ are $\mathbb{R}$-linear operators such that $\partial_s^{-1} f \in A_0$ and $\int f \in A_0$ for all $f \in A_2$,

(a7) $\kappa \in A_1, \kappa^{-1} \in A_{-1}$, and $\tau, s, l, 1 \in A_0$,

(b1) ∂_s is a derivation of $A_{-\infty}$, i.e., Eq. (2) hold for all $f, g \in A_{-\infty}$,

(b2) Eqs. (6) and (7) hold for all $f \in A_2$ and $F \in \operatorname{Ker} \partial_s$,

(b3) $\int (f \partial_s^{-1} g) = - \int (g \partial_s^{-1} f)$ for all $f, g \in A_2$,

(b4) $\kappa\kappa^{-1} = \partial_s s = 1$, and $\partial_s l = 0$.

Let us consider the objects $\mathcal{E}_n$ that are fully characterized by the rules (a1)–(a7) above; regarding (a1)–(a7) (in which A_n should be read as $\mathcal{E}_n$) as the axioms for $\mathcal{E}_n$, we define $\mathcal{E}_n$, $n \in \mathbb{Z}$, as a family of objects generated by the symbols or indeterminates $\{\kappa, \kappa^{-1}, \tau, s, l, 1\}$ with the algebraic operations. Here and in the following paragraph, by algebraic operations we mean addition, scaling by a real number, multiplication, ∂_s, ∂_s^{-1} and $\int$. It is the implication of (a6) that ∂_s^{-1} and $\int$ cannot act on $\mathcal{E}_n$, $n < 2$. By definition, rules (such as (b1)–(b4)) not following from (a1)–(a7) are not available for $\mathcal{E}_n$.

Let $g_1, \ldots, g_r$ be independent variables running over $\mathcal{E}_{n_1}, \ldots, \mathcal{E}_{n_r}$, respectively. We say $f(g_1, \ldots, g_r)$ is an $\mathcal{E}_n$-valued variable algebraically depending on $g_1, \ldots, g_r$ if $f(g_1, \ldots, g_r)$ is an expression written in terms of $\{g_1, \ldots, g_r, \kappa, \kappa^{-1}, \tau, s, l, 1\}$ with use of the algebraic operations and if the rules (a1)–(a7) supplemented with the condition $g_i \in \mathcal{E}_{n_i}$ conclude $f(g_1, \ldots, g_r) \in \mathcal{E}_n$.

We denote by $\mathcal{A}_n$ the space of scalar fields on BAL having an expression that belongs to $\mathcal{E}_n$. It is easy to see that $\mathcal{A}_n$ is a subset of A_n. The statements (a1)–(a7) and (b1)–(b4) remain true even if every A_n is read as $\mathcal{A}_n$. Moreover, these together with the positive-definiteness or at least non-degeneracy of the bi-linear form (35) are all of the fundamental setting we need in constructing the theory developed in this paper.

Proposition 1. *Let Φ be an $\mathbb{R}$-linear map $\mathcal{A}_1 \to \mathcal{A}_0$ induced from an $\mathbb{R}$-linear map $\mathcal{E}_1 \to \int(\mathcal{E}_2)$ in the apparent way. If this map is written as $\Phi(g) = \int f(g)$ with an $\mathcal{E}_2$-valued variable $f(g)$ algebraically depending on $g \in \mathcal{E}_1$, then there exists $h \in \mathcal{A}_1$ with which one can write $\Phi(g) = \int gh \; \forall g \in \mathcal{A}_1$ as an equation in $\mathcal{A}_0$.*

Proof. We note the formulae

$$\int (f\partial_s g) = -\int (g\partial_s f), \quad \int (f\partial_s^{-1} g) = -\int (g\partial_s^{-1} f),$$
$$\int \left(f\int g\right) = \int \left(g \int f\right), \tag{22}$$

each of which is valid as an equation in $\mathcal{A}_0$ if the left-hand side is given as an $\mathcal{E}_0$-valued variable algebraically depending on f and g. From bi-$\mathbb{R}$-linearity of multiplication and $\mathbb{R}$-linearity of ∂_s, ∂_s^{-1} and $\int$, we see the existence of an expression $\Phi(g) = \sum_i \int f_i(g)$ with $f_i(g)$ being $\mathcal{E}_2$-valued variables algebraically depending on $g \in \mathcal{E}_1$ such that no additions are used in the expression of $f_i(g)$. Further, it is possible to suppose g appears in each expression $\int f_i(g)$ only once because of the $\mathbb{R}$-linearity of Φ. For such expressions, it is apparent how to apply successively formulae (22) to $\int f_i(g)$ to rewrite it into the form $\int gh_i$. This process is justified if one considers the equations in $\mathcal{A}_0$, while consideration in $\mathcal{E}_0$ is useful in verifying that the expressions $\int(\cdots)$ appearing in each step make sense as $\mathcal{E}_0$-valued variables and eventually in deducing $h_i \in \mathcal{A}_1$. □

Let $\mathcal{T}_n, n \in \mathbb{Z}$, be the $\mathbb{R}$-vector space of tangent fields defined by $\mathcal{T}_n := \{f\mathbf{t}+g\mathbf{n}+h\mathbf{b} \mid f \in \mathcal{A}_{n-1}, g, h \in \mathcal{A}_n\}$. It is easy to see that $\mathcal{T}_n$ is ∇_s-invariant, i.e., $\nabla_s(\mathcal{T}_n) \subset \mathcal{T}_n$.

We denote by φ the surjection associated with the identification $f\mathbf{t} \sim 0$ in $\mathcal{T}_n$, namely, putting $N = \varphi(\mathbf{n})$, $B = \varphi(\mathbf{b})$, we write

$$\varphi(f\mathbf{t} + g\mathbf{n} + h\mathbf{b}) = gN + hB \tag{23}$$

for scalar fields f, g, h. The vector spaces $\varphi(\mathcal{T}_n)$ are left $\mathcal{A}_0$-modules with $f(gN + hB) = (fg)N + (fh)B \ \forall f \in \mathcal{A}_0, \forall g, h \in \mathcal{A}_n$.

Let $\mathcal{X} := \varphi(\mathcal{T}_1) = \{gN + hB \mid g, h \in \mathcal{A}_1\}$. Each element of $\mathcal{X}$ is referred to as a vector field on BAL. Through the injection $\wp : \mathcal{X} \to \mathcal{T}_1$ defined by

$$\wp(gN + hB) := \partial_s^{-1}(\kappa g)\mathbf{t} + g\mathbf{n} + h\mathbf{b}, \tag{24}$$

a vector field X induces a derivation – variational differential associated with $\wp(X)$. This derivation acting on $\mathcal{A}_n$ or $\mathcal{T}_n$ can be evaluated with formulae (11)–(20).

Proposition 2. *The vector spaces $\mathcal{A}_n$ and $\mathcal{T}_n$ are invariant under the action of the vector fields, namely, $\delta_{\wp(X)}(\mathcal{A}_n) \subset \mathcal{A}_n$ and $\nabla_{\wp(X)}(\mathcal{T}_n) \subset \mathcal{T}_n \ \forall X \in \mathcal{X}, \forall n \in \mathbb{Z}$.*

Proof. It is essential that every vector field $X = gN + hB$ is written with $g, h \in \mathcal{A}_1$. Taking notice of this situation, we find that formulae (11)–(20) ensure the invariance of $\mathcal{A}_n$ and $\mathcal{T}_n$ under the action of vector fields. □

The space $\mathcal{X}$ of vector fields is a Lie algebra, and $\mathcal{A}_n$ are $\mathcal{X}$-modules. This is an immediate consequence of the following theorem.

Theorem 3. *The $\mathbb{R}$-vector space $\wp(\mathcal{X})$ is a Lie algebra with the product*

$$[\mathbf{x}, \mathbf{y}] := \nabla_{\mathbf{x}}\mathbf{y} - \nabla_{\mathbf{y}}\mathbf{x} \quad \forall \mathbf{x}, \mathbf{y} \in \wp(\mathcal{X}). \tag{25}$$

For every $n \in \mathbb{Z}$, the algebra $\mathcal{A}_n$ is a $\wp(\mathcal{X})$-module with the action $\delta_{\mathbf{x}}$, $\mathbf{x} \in \wp(\mathcal{X})$, namely,

$$(\delta_{\mathbf{x}}\delta_{\mathbf{y}} - \delta_{\mathbf{y}}\delta_{\mathbf{x}} - \delta_{[\mathbf{x},\mathbf{y}]})f = 0 \quad \forall \mathbf{x}, \mathbf{y} \in \wp(\mathcal{X}), \quad \forall f \in \mathcal{A}_n. \tag{26}$$

Proof. The statements are verified by using (11)–(20). A convenient procedure is as follows: First, verify that $[\mathbf{x}, \mathbf{y}]$ belongs to $\wp(\mathcal{X})$ for all $\mathbf{x}, \mathbf{y} \in \wp(\mathcal{X})$. Second, show Eq. (26) holds for $f = \kappa, \tau, s, l$ and then extend (26) to the whole $\mathcal{A}_{-\infty} = \mathcal{A}_0 \cup \mathcal{A}_{-1} \cup \cdots$ Finally, verify the Jacobi identity in $\wp(\mathcal{X})$ with the help of (26). □

Theorem 3 is quite similar to Theorem 1 of [14] in particular in the proof, though the considered objects are different.

To simplify expressions, we put

$$\tilde{\nabla}_X := \varphi \circ \nabla_{\wp(X)} \circ \wp \tag{27}$$

$\forall X \in \mathcal{X}$, so that we can write the commutator product of $\mathcal{X}$ as

$$[X, Y] = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X \quad \forall X, Y \in \mathcal{X}. \tag{28}$$

Likewise we put

$$\langle g_1 N + h_1 B, g_2 N + h_2 B\rangle_\perp := g_1 g_2 + h_1 h_2, \tag{29}$$

which defines a bi-linear form $\varphi(\mathcal{T}_i) \times \varphi(\mathcal{T}_j) \to \mathcal{A}_{i+j}$. Then, we have

$$\begin{aligned}\tilde{\nabla}_X Y = {}& (\delta_{\wp(X)}\langle N, Y\rangle_\perp)N + (\delta_{\wp(X)}\langle B, Y\rangle_\perp)B + (\partial_s^{-1}\langle \kappa N, Y\rangle_\perp)\varphi\nabla_s\wp(X) \\ & - \langle \kappa^{-1}B, \varphi\nabla_s\nabla_s\wp(X)\rangle_\perp(\langle B, Y\rangle_\perp N - \langle N, Y\rangle_\perp B)\end{aligned} \tag{30}$$

for all $X, Y \in \mathcal{X}$.

Let $\mathcal{F}$ be the subalgebra of $\mathcal{A}_0$ generated by 1, l and the elements of $\int(\mathcal{A}_2)$. The vector space $\mathcal{F}$ is an $\mathcal{X}$-submodule of $\mathcal{A}_0$, i.e., $\delta_{\wp(X)}(\mathcal{F}) \subset \mathcal{F}$ $\forall X \in \mathcal{X}$. We denote the action of $\mathcal{X}$ on $\mathcal{F}$ by the left-action, namely, $XF = \delta_{\wp(X)} F$ $\forall X \in \mathcal{X}, \forall F \in \mathcal{F}$. This action is a derivation:

$$X(FG) = (XF)G + F(XG) \quad \forall X \in \mathcal{X}, \ \forall F, G \in \mathcal{F}. \tag{31}$$

Since $\mathcal{F} \subset \mathcal{A}_0$, we see that $\mathcal{X}$ is a left $\mathcal{F}$-module. Taking notice of $\partial_s F = 0$ $\forall F \in \mathcal{F}$ and referring (11)–(20), we easily find $\delta_{\wp(FX)}\, g = F\delta_{\wp(X)} g$, $\tilde{\nabla}_{FX} Y = F\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X(FY) = (\delta_{\wp(X)} F)Y + F\tilde{\nabla}_X Y$ $\forall F \in \mathcal{F}, \ \forall X, \ Y \in \mathcal{X}, \ \forall g \in \mathcal{A}_n$. Hence we see

$$(FX)G = F(XG) \quad \forall F, G \in \mathcal{F}, \ \forall X \in \mathcal{X}, \tag{32}$$

$$\mathcal{L}_X(FY) = (\mathcal{L}_X F)Y + F\mathcal{L}_X Y \quad \forall X, Y \in \mathcal{X}, \ \forall F \in \mathcal{F}, \tag{33}$$

where

$$\mathcal{L}_X Y := [X, Y], \quad \mathcal{L}_X F := XF \quad \forall X, Y \in \mathcal{X}, \ \forall F \in \mathcal{F}. \tag{34}$$

Below, we construct in the usual, algebraic manner a differential calculus, in which the algebra $\mathcal{F}$ consisting of functionals on BAL plays the role of the algebra of functions. The construction is based on the pair $(\mathcal{F}, \mathcal{X})$ of commutative algebra and Lie algebra. It is essential for this construction that $\mathcal{F}$ is a left $\mathcal{X}$-module, $\mathcal{X}$ is a left $\mathcal{F}$-module, and the Eqs. (31)–(33) hold. We would like to make a further remark. Let $g : \mathcal{X} \times \mathcal{X} \to \mathcal{F}$ be a symmetric form defined by

$$g(X, Y) := \int \langle X, Y \rangle_{\perp} \tag{35}$$

with (29). This is bi-$\mathcal{F}$-linear and positive-definite. We refer to g as the Riemannian structure on BAL. Given $F \in \mathcal{F}$, the vector field $X \in \mathcal{X}$ such that $YF = g(X, Y)\ \forall Y \in \mathcal{X}$ is called the gradient of F and is denoted by grad F. The existence of the gradient for every element of $\mathcal{F}$ can be verified by virtue of Propositions 1 and 2. In contrast to the differential calculus on finite-dimensional Riemannian manifolds, this seems to be quite non-trivial. This situation is necessary for realizing the space of 1-forms as a space identifiable with $\mathcal{X}$.

Let $\mathcal{D}^p$, $p = 1, 2, \ldots$, denote the vector space of maps $\eta : \mathcal{X}^{\times p} \to \mathcal{F}$ such that $F := \eta(U_1, \ldots, U_p)$ with $U_i = g_i N + h_i B \in \mathcal{X}$ is $\mathcal{F}$-linear in each U_i, skew-symmetric (if $p \geq 2$) under the exchange of U_i and U_j, $i \neq j$, and expressible as an $\mathcal{E}_0$-valued variable of the form $F = \int(\cdots)$ algebraically depending on $g_1, h_1, \ldots, g_p, h_p$. Such a map $\eta \in \mathcal{D}^p$ is referred to as a pth order differential form or p-form on BAL. For the case $p = 0$, we define $\mathcal{D}^0 := \mathcal{F}$. From Proposition 1 and the non-degeneracy of (35), we see that for every 1-form ξ there uniquely exists a vector field X such that $\xi(Y) = g(X, Y)\ \forall Y \in \mathcal{X}$.

The exterior derivative is a map $\mathrm{d} : \mathcal{D}^p \to \mathcal{D}^{p+1}$ defined by

$$\begin{aligned}(\mathrm{d}\eta)&(U_0, \ldots, U_p)\\ &= \sum_{i=0}^{p} (-1)^i U_i(\eta(U_0, \ldots, \check{U}_i, \ldots, U_p))\\ &\quad + \sum_{i<j} (-1)^{i+j} \eta([U_i, U_j], U_0, \ldots, \check{U}_i, \ldots, \check{U}_j, \ldots, U_p)\\ &\quad \forall \eta \in \mathcal{D}^p,\ \forall U_i \in \mathcal{X},\end{aligned} \tag{36}$$

where $\check{U}_i$ stands for the absense of U_i. This is a coboundary operator, i.e., $\mathrm{d} \circ \mathrm{d} = 0$. The interior product $\iota_X : \mathcal{D}^{p+1} \to \mathcal{D}^p$ with $X \in \mathcal{X}$ is defined by

$$(\iota_X \eta)(U_1, \ldots, U_p) = \eta(X, U_1, \ldots, U_p). \tag{37}$$

The Lie derivative $\mathcal{L}_X$ in the direction of $X \in \mathcal{X}$ is an operator acting on each $\mathcal{X}$-module consisting of certain $\mathcal{F}$-vectors, so-called tensor fields, and $X \mapsto \mathcal{L}_X$ provides a representation of the Lie algebra $\mathcal{X}$. The definition was already given both on $\mathcal{F}$ and $\mathcal{X}$ in (34). The extension to other tensor fields is made by imposing Leibnitz rule. For example, if R is an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$, then $\mathcal{L}_X(RY) = (\mathcal{L}_X R)Y + R\mathcal{L}_X Y\ \forall X, Y \in \mathcal{X}$. For a p-form η, the formula

$$\mathcal{L}_X \eta = \iota_X \mathrm{d}\eta + \mathrm{d}\iota_X \eta \tag{38}$$

is available. It is possible to introduce the exterior product, which is, however, not used in this paper.

4. Structure related to the integrability

We begin this section with referring to two works [4,5].

Let J be the operator $\mathcal{X} \to \mathcal{X}$,

$$J(X) = \langle B, X\rangle_\perp N - \langle N, X\rangle_\perp B. \tag{39}$$

The operator induces a Poisson structure [4], $\{F, G\} = (J \operatorname{grad} F)\, G\ \forall F, G \in \mathcal{F}$, with which the vortex filament equation (1) can be understood as a Hamiltonian equation if l is chosen to be the Hamiltonian functional; indeed

$$\kappa \mathbf{b} = \wp(J \operatorname{grad} l). \tag{40}$$

Making use of the Hasimoto map, Langer and Perline [5] found that the vector fields $\kappa B, KJ^{-1}(\kappa B), (KJ^{-1})^2(\kappa B), \ldots$ are Hamiltonian flows associated with the constants of motion in involution, where K is defined by

$$K(X) = J\varphi\nabla_s\wp J(X). \tag{41}$$

The operator

$$R := KJ^{-1} = J \circ \varphi \circ \nabla_s \circ \wp \tag{42}$$

is referred to as the recursion operator.

It should be emphasized that the definition of $\mathcal{X}$ given in the preceding section is consistent with J, K and R, namely, these operators make sense as $\mathcal{F}$-linear maps $\mathcal{X} \to \mathcal{X}$.

The following is our fundamental result regarding the integrability of BAL.

Theorem 4. *The recursion operator R defined by* (42) *is hereditary.*

Before giving the proof, we recall that for an arbitrary $\mathcal{F}$-linear operator $R: \mathcal{X} \to \mathcal{X}$ the term hereditary [6,9] (see also [7,8,11,13]) means absence of the Nijenhuis torsion N_R,

$$\begin{aligned} N_R(X, Y) &:= (\mathcal{L}_{RX}R - R\mathcal{L}_X R)Y \\ &= [RX, RY] - R[RX, Y] - R[X, RY] + R^2[X, Y]. \end{aligned} \tag{43}$$

Proof. For all $X \in \mathcal{X}$, define $\tilde{\nabla}_X R : \mathcal{X} \to \mathcal{X}$ by

$$(\tilde{\nabla}_X R)(Y) = \tilde{\nabla}_X(RY) - R(\tilde{\nabla}_X Y) \quad \forall Y \in \mathcal{X}, \tag{44}$$

which is found to be

$$(\tilde{\nabla}_X R)(Y) = (\partial_s^{-1}\langle \kappa N, RY\rangle_\perp) J^{-1}RX + (\partial_s^{-1}\langle RX, RY\rangle_\perp)\kappa B \tag{45}$$

by using (30). Substituting the formula above into

$$N_R(X, Y) = (\tilde{\nabla}_{RX} R)(Y) - R((\tilde{\nabla}_X R)(Y) - (X \leftrightarrow Y),$$

we find $N_R = 0$. □

We remark that the operator J, $J^2 = -1$, is a complex structure; it can be shown by a direct calculation that the Nijenhuis torsion of J vanishes, though this fact is not used in this paper.

Since J and K are skew-adjoint, i.e., $g(JX, Y) = -g(X, JY)$, $g(KX, Y) = -g(X, KY)$, we can define 2-forms Ω_n, $n = 0, 1, \dots$ with

$$\Omega_n(X, Y) = \Omega_0(R^n X, Y), \quad \Omega_0(X, Y) = g(J^{-1}X, Y) \qquad \forall X, Y \in \mathcal{X}. \tag{46}$$

It is possible to show that $\mathrm{d}\Omega_0 = \mathrm{d}\Omega_1 = 0$ by using (30). Since R is hereditary, this extends (see Theorem 3.9 in [13] or use (A.3) in Appendix A) to

$$\mathrm{d}\Omega_0 = \mathrm{d}\Omega_1 = \mathrm{d}\Omega_2 = \cdots = 0. \tag{47}$$

We note that $\mathrm{d}\Omega_0 = 0$ is implied in [4], because Ω_0 is the symplectic structure corresponding to the Marsden–Weinstein Poisson structure.

As a consequence of (47), in addition to the hereditary property of R, the operators J and K are found to form a Hamiltonian pair, a structure implying the integrability [13]. To see this, apply Theorem A.1 in Appendix A to the present case with reading ω as Ω_0 and understand the identification caused by the Riemannian structure g, making H_0 and H_1 equivalent to J and K, respectively.

Another fundamental result that we obtained is of master symmetries [6,10]. Let us introduce two sequences of vector fields,

$$X_n = R^n(\kappa B), \quad n = 0, 1, 2, \dots, \tag{48}$$

$$Y_n = R^{n-1}(s\kappa B), \quad n = 1, 2, 3, \dots \tag{49}$$

The vector fields X_n are those given in [5] with a difference in their index (shifted by 2).

The vector fields $X_0, X_1, \dots$ and $Y_1, Y_2, \dots$ form a Lie subalgebra of $\mathcal{X}$ such that

$$[X_n, X_m] = 0, \tag{50}$$

$$[X_n, Y_m] = (n + 2)X_{n+m}, \tag{51}$$

$$[Y_n, Y_m] = (n - m)Y_{n+m}. \tag{52}$$

To obtain these equations, we need to verify

$$\mathcal{L}_{X_0} R = 0, \quad \mathcal{L}_{Y_1} R = -R^2, \quad [X_0, Y_1] = 2X_1 \tag{53}$$

by direct calculation, e.g., writing $(\mathcal{L}_X R)(Y) = (\tilde{\nabla}_X R)(Y) - \tilde{\nabla}_{RY} X + R\tilde{\nabla}_Y X$ and applying (30) and (45). Then, all Eqs. (50)–(52) follow, since as a consequence of the hereditary property of R there exists the identity

$$[R^m X, R^n Y] = -R^n(\mathcal{L}_Y R^m)X + R^m(\mathcal{L}_X R^n)Y + R^{m+n}[X, Y] \quad \forall X, Y \in \mathcal{X}.$$

See e.g., [6,13] for the general consideration.

Since X_0 is the flow of the vortex filament equation, commuting flows X_n represent symmetries of the equation. These symmetries are generated from X_0 by the action of Y_n as in (51); vector fields Y_n provide master symmetries [10].

The vector fields X_n and Y_n act on the 2-forms Ω_n and the 1-forms $\zeta_n := \iota_{X_n}\Omega_0$ in the following way:

$$\mathcal{L}_{X_{m-1}}\Omega_n = 0, \quad \mathcal{L}_{Y_m}\Omega_n = (3-m-n)\Omega_{m+n}. \tag{54}$$

$$\mathcal{L}_{X_{m-1}}\zeta_n = 0, \quad \mathcal{L}_{Y_m}\zeta_n = (1-m-n)\zeta_{m+n}, \tag{55}$$

For the proof of (54), derive it in the case $m = 1, n = 0$ by direct calclation, and then use the hereditary property of R and the fact $\mathrm{d}\Omega_n = 0$. Then, (55) follows from (54) and the fact $\mathrm{d}\zeta = 0$, which also is a consequence of (54).

It is a remarkable consequence of (54) and (55) that the 2-forms Ω_n, $n = 1, 2, 4, 5, \ldots$ and the 1-forms ζ_n, $n = 2, 3, \ldots$ are exact. Indeed, $\Omega_n = \mathrm{d}\iota_{Y_1}\Omega_{n-1}/(3-n)$ and $\zeta_n = \mathrm{d}\iota_{Y_1}\zeta_{n-1}/(1-n)$. From the latter, we can read the expression $I_n = \zeta_{n-1}(Y_1)/(1-n)$, $n = 2, 3, \ldots$ of the constants of motion in involution. With some arguments, this expression is found to agree with the inspection of Langer–Perline [5], see also [14]. We also remark that essentially the same method deriving the expression of constants of motion in involution is given by Dorfman (see Theorem 7.10 in [13]) in much more general setting.

Appendix A. Hereditary operator and Hamiltonian pair

A hereditary operator and a Hamiltonian pair are closely related to each other. Detailed analysis for this situation can be found in [13]. We, however, present here again a part of the results relevant for the present paper. We would like to mention that a generalization – minor but requisite for our purpose – is achieved here; in Theorem A.1 below, we need not suppose the operator R is invertible.

Let R be an $\mathcal{F}$-linear map $\mathcal{X} \to \mathcal{X}$. If a p-form ω, $p \geq 1$, satisfies the condition

$$\iota_{RY}\iota_X\omega = \iota_Y\iota_{RX}\omega \quad \forall X, Y \in \mathcal{X}, \tag{A.1}$$

then p-forms ω_n, $n = 0, 1, 2, \ldots$, such that

$$\iota_X\omega_n = \iota_{R^nX}\omega \quad \forall X \in \mathcal{X} \tag{A.2}$$

can be introduced. The p-forms ω_n introduced in this way obey the same condition as in (A.1). Throughout the appendix, we suppose that p-forms having an index are all introduced as above for a given $\mathcal{F}$-linear map R and the index runs over non-negative integers.

We begin with preparation of several formulae. The following (in case ω_n are 2-forms) is given in Proposition 3.8 of [13]:

$$\begin{aligned}&\iota_Y\iota_X\,\mathrm{d}\omega_{n+2} - \iota_{RY}\iota_X\,\mathrm{d}\omega_{n+1} - \iota_Y\iota_{RX}\,\mathrm{d}\omega_{n+1} + \iota_{RY}\iota_{RX}\,\mathrm{d}\omega_n\\ &\quad = -\iota_{N_R(X,Y)}\omega_n \quad \forall X, Y \in \mathcal{X},\end{aligned} \tag{A.3}$$

where N_R is the Nijenhuis torsion (43) of R. To see this, apply the identity $\iota_Y\iota_X\,\mathrm{d} = \iota_Y\mathcal{L}_X - \mathcal{L}_Y\iota_X + \mathrm{d}\iota_Y\iota_X$ to the left-hand side of (A.3), and then use the Leibnitz rule $\iota_Y\mathcal{L}_X\omega_{m+1} = \iota_{RY}\mathcal{L}_X\omega_m + \iota_{(\mathcal{L}_XR)Y}\omega_m$ of Lie derivative.

The next formula is an equation for 2-forms. Suppose ω is a 2-form satisfying (A.1). Then,

$$\begin{aligned}
&\{(\mathcal{L}_{R^m X}\omega_n)(Y, Z) - \omega_{m+n}(X, [Y, Z])\} + \text{cycle} \\
&\quad = \{\mathrm{d}\omega_n(R^m X, Y, Z) + X(\omega_{m+n}(Y, Z)) - \omega_{m+n}([X, Y], Z) + \text{cycle}\} \\
&\qquad - 3\,\mathrm{d}\omega_{m+n}(X, Y, Z) \\
&\quad = \{\mathrm{d}\omega_n(R^m X, Y, Z) + \text{cycle}\} - 2\,\mathrm{d}\omega_{m+n}(X, Y, Z) \quad \forall X, Y, Z \in \mathcal{X},
\end{aligned} \tag{A.4}$$

where cycle stands for the cyclic permutation in X, Y, Z. The first equality of (A.4) can be verified with the help of the formula $\mathcal{L}_{R^n X}\omega_m = \iota_{R^n X}\,\mathrm{d}\omega_m + \mathcal{L}_X\omega_{m+n} - \iota_X\,\mathrm{d}\omega_{m+n}$ $\forall X \in \mathcal{X}$, which is an immediate consequense of (38). The second equality of (A.4) follows from the definition (36) of exterior derivative.

Let us consider the case in which the map $\mathcal{X} \to \mathcal{D}^1$, $X \mapsto \iota_X\omega$ is invertible, so that an $\mathcal{F}$-linear map $H_0 : \mathcal{D}^1 \to \mathcal{X}$ is defined by $\omega(H_0\xi, Y) = \xi(Y)$ $\forall \xi \in \mathcal{D}^1$, $\forall Y \in \mathcal{X}$. We put $H_n := R^n H_0$, $n = 0, 1, 2, \ldots$ To summarize,

$$\omega(H_n\xi, Y) = \xi(R^n Y) \quad \forall \xi \in \mathcal{D}^1, \quad \forall Y \in \mathcal{X}. \tag{A.5}$$

It is easy to see that H_n are skew-symmetric, i.e., $\xi(H_i\eta) = -\eta(H_i\xi)$ $\forall \xi, \eta \in \mathcal{D}^1$. The following is a generalization of Theorem 3.13 in [13].

Theorem A.1. Let $R : \mathcal{X} \to \mathcal{X}$ be a hereditary operator and ω a 2-form satisfying (A.1) and $\mathrm{d}\omega_0 = d\omega_1 = 0$ with (A.2). If $H_0, H_1, H_2, \ldots$ are $\mathcal{F}$-linear maps $\mathcal{D}^1 \to \mathcal{X}$ satisfying (A.5), then H_m and H_n form a Hamiltonian pair.

Before giving the proof, let us briefly review the notion of a Hamiltonian operator/pair [7,8]. For two skew-symmetric $\mathcal{F}$-linear maps $H_m, H_n : \mathcal{D}^1 \to \mathcal{X}$, the Schouten bracket $[H_m, H_n] : \mathcal{D}^1 \times \mathcal{D}^1 \times \mathcal{D}^1 \to \mathcal{F}$ can be introduced [8]. The definition can be written as

$$\begin{aligned}
[H_m, H_n](\xi, \eta, \zeta) = &\{-(H_m\eta)(\zeta(H_n\xi)) + \zeta([H_m\eta, H_n\xi])\} \\
&+ (m \leftrightarrow n) + \text{cycle}(\xi, \eta, \zeta).
\end{aligned} \tag{A.6}$$

A skew-symmetric $\mathcal{F}$-linear map $H : \mathcal{D}^1 \to \mathcal{X}$ is referred to as a Hamiltonian operator if $[H, H] = 0$. A Hamiltonian operator H induces the Poisson structure $\{F, G\}_H = (H\,\mathrm{d}F)G$ $\forall F, G \in \mathcal{F}$. The map $H \circ \mathrm{d}$ associating the functionals with the Hamiltonian vector fields is a morphism $\mathcal{F} \to \mathcal{X}$ of Lie algebras. Two Hamiltonian operators H_m and H_n are said to form a Hamiltonian pair if $[H_m, H_n] = 0$.

To prove Theorem A.1, we show that the equation

$$\begin{aligned}
&[H_m, H_n](\xi, \eta, \zeta) \\
&\quad = 4\mathrm{d}\omega_{m+n}(H_0\xi, H_0\eta, H_0\zeta) \\
&\quad -\{\mathrm{d}\omega_m(H_n\xi, H_0\eta, H_0\zeta) + (m \leftrightarrow n) + \text{cycle}(\xi, \eta, \zeta)\}
\end{aligned} \tag{A.7}$$

is held if R is a hereditary operator. Using the relation $[R^m X, R^n Y] - R^n[R^m X, Y] = R^m[X, R^n Y] - R^{m+n}[X, Y]\ \forall X, Y \in \mathcal{X}$ (the hereditary property), we can rewrite (A.6) into

$$\begin{aligned}[H_m, H_n](\xi, \eta, \zeta) = \{&-(H_m\eta)(\omega_n(H_0\zeta, H_0\xi)) + \omega_n(H_0\zeta, [H_m\eta, H_0\xi]) \\ &+ \omega_m(H_0\zeta, [H_0\eta, H_n\xi]) - \omega_{m+n}(H_0\zeta, [H_0\eta, H_0\xi])\} \\ &+ (m \leftrightarrow n) + \text{cycle}(\xi, \eta, \zeta).\end{aligned}$$

Taking notice of the symmetry under the exchange $m \leftrightarrow n$ and the cyclic permutation, we see that the first three terms in the right-hand side sum up to $(-\mathcal{L}_{H_m\xi}\omega_n)(H_0\eta, H_0\zeta)$. Then, with the help of (A.4) we obtain (A.7).

Since R is hereditary, the right-hand side of (A.3) vanishes. Hence $d\omega_0 = d\omega_1 = 0$ asserts $d\omega_n = 0$ for all non-negative integers n. Therefore, the right-hand side of (A.7) also vanishes and we obtain Theorem A.1.

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